Nonuniversality and scaling breakdown in a nonconservative earthquake model

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We use extensive numerical simulations to test recent claims of universality in the nonconservative regime of the Olami-Feder-Christensen model. By studying larger systems and a wider range of dissipation levels than previously considered we conclude that there is no evidence of universality in the model with only limited regions of the event size distributions displaying power-law behavior. We further analyze the dimension of the largest events in the model, $D_{\max}$, using a multiscaling method. This reveals that although $D_{\max}$ initially increases with system size, for larger systems the dimension ultimately decreases with system size casting further doubt on the criticality of the model.

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I. INTRODUCTION

There has been a lot of interest recently in the theory of self-organized criticality (SOC) [1] which provides one explanation for the wealth of scale invariant behavior observed in nature. Self-organized criticality was introduced in 1987 by Bak, Tang, and Wiesenfeld (BTW) [2] in the context of a simple sandpile model. One major drawback of the BTW model is that criticality can only be obtained if the system variables are conserved [3]. Unfortunately, conservation is not natural in many physical systems such as earthquakes and landslides, therefore models allowing some dissipation must be introduced.

There are a number of dissipative or nonconservative models, with the most studied being the Olami-Feder-Christensen (OFC) model [4] which is based on the Burridge-Knopoff spring-block model for earthquakes [5]. In two dimensions the OFC model is defined on a square lattice with $L\times L$ sites, where each node (i,j) is initially assigned with a random variable or energy u_{ij}, in the interval $[0,1)$. The system is then slowly driven in such a way that the energy at all the sites increases uniformly until one site reaches the threshold value $u_{th}=1$. Once the energy on a site reaches the threshold value the site is termed supercritical, and the system then undergoes a relaxation process where energy is redistributed to neighboring sites. The rules for relaxation require the supercritical site to relax according to $u_{ij}\rightarrow 0$, and its supercritical energy to be redistributed to (typically) four neighbors, u_{nn}, using the rule $u_{nn}\rightarrow u_{nn}+\alpha u_{ij}$. This toppling is repeated until all sites in the system are below the threshold value, after which the driving phase recommences until the next event is triggered. These toppling events are called "avalanches" which can be considered to represent earthquakes, with the size s of the earthquake given by the number of topples in the avalanche. The parameter α used in the redistribution rule controls the amount of energy dissipated in the system. When $\alpha=0.25$ energy is conserved, but when $\alpha<0.25$ some energy is lost so the OFC model represents a nonconservative system.

The OFC model was originally proposed to display SOC behavior [4], where one identifies criticality with scale invariance of the event size distribution so that the critical state is characterized by the lack of any typical length scale in an infinite-sized system. For such a critical state one expects to find a power-law distribution of earthquake sizes. OFC indeed found an approximately power-law distribution when running simulations on finite lattice sizes, with additional evidence that finite-size scaling (FSS) was satisfied. The FSS ansatz for the probability distribution of earthquake sizes in a system of size L (i.e., a lattice of $L\times L$ sites), $P_L(s)$, is

$$P_L(s)\sim L^{-\beta}G\left(\frac{s}{L^D}\right), \tag{1.1}$$

where G is a suitable scaling function and β and D are critical exponents describing the scaling of the distribution function. The ratio $\tau=\beta/D$ is the power-law exponent for the infinite-system size probability distribution $P(s)\sim s^{-\tau}$ and is predicted to be dependent on α [4]. In their investigations OFC found that the exponential cutoff scaled with $D>2$, which has since been criticized as physical constraints require $D\leqslant 2$ for large enough system sizes [6,7].

Later studies by Grassberger [8], where larger system sizes were used in simulations, supported the fact that the OFC model was critical but concluded that FSS is violated. The observation of FSS in earlier studies is believed to be due to the small system sizes considered, although a recent analysis suggests that the probability distribution for events starting in the bulk of the system may be compatible with the FSS hypothesis [9]. Further studies examining the branching rate in the OFC model have concluded that the system is only truly critical in the conservative case $\alpha=0.25$ with the model being "almost critical" for large but nonconservative choices of α [10,11], however these findings remain controversial [7,12].

Recently, Lise and Paczuski (LP) [7] have reanalyzed the OFC model using a simple multiscaling ansatz. Contrary to earlier studies, they found evidence of universal behavior, that is, they found that the slope exponent τ is independent of α (for a wide range of α values at least), with $\tau=1.8$. Once again FSS is found to be violated, but for the larger system sizes LP considered they found $D\leqslant 2$ is satisfied. By applying a multiscaling approach they identified the dimension of the largest avalanche in the system, $D_{\max}$, and found evidence that $D_{\max}\to 2$ as $L\to\infty$ for a range of dissipation levels. Such results indicate that the nonconservative OFC model is critical. In this paper we reexamine the findings of LP considering larger system sizes L, and a wider range of α values leading us to different conclusions.

The remainder of this paper is organized as follows. In Sec. II we examine the event size distribution for a range of α values, revealing that there is no evidence for a universal exponent τ. Indeed, for $\alpha<0.25$ we show that the probability distribution cannot be fit by a single power law in general, so that the systems do not display genuine scale invariance. In Sec. III we investigate the claim that $D_{\max}\to 2$ as L increases and present evidence that although $D_{\max}$ initially increases towards 2, when the system size is sufficiently large $D_{\max}$ decreases as L increases. We discuss our results and present our conclusions in Sec. IV.

II. NONUNIVERSALITY OF EVENT SIZE DISTRIBUTIONS

Using numerical simulations we have investigated the event size distributions for a wide range of dissipation levels. Following LP we use a multiscaling analysis to assist us in obtaining clearer results. The multiscaling ansatz used for the probability distribution function $P_L(s)$ takes the form

$$\frac{\log(P_L(s))}{\log(L/l_0)}=F\left(\frac{\log(s/s_0)}{\log(L/l_0)}\right), \tag{2.1}$$

where formally s_0 and l_0 are parameters which are chosen to obtain the best data collapse for different system sizes [13]. For our study we again follow LP and simplify Eq. (2.1) by choosing $s_0=l_0=1$. For a given α, if $P_L(s)\sim s^{-\tau(\alpha)}$ for some range of event sizes, then $\log(P_L)/\log(L)\sim-\tau(\alpha)\log(s)/\log(L)$ in that range. This shows that the curve $\log(P_L)/\log(L)+\tau(\alpha)D_{av}$, where $D_{av}=\log(s)/\log(L)$ is the "avalanche dimension", should converge to a horizontal straight line for sufficiently large L if the probability distribution is indeed power law [14]. The value of $\tau(\alpha)$ that gives the horizontal line is precisely the slope exponent discussed in the Introduction. In this paper we aim to test the prediction that for nonconservative systems the slope exponent takes a universal value of 1.8. Thus we present our data as plots of $\log(P_L)/\log(L)+1.8D_{av}$ versus D_{av} for a range of system sizes. If the prediction is correct, the curves should converge to a horizontal line as system size increases, and if the curves converge to a straight line with a slope p, say, then the original data were power law with the asymptotic distribution $P(s)\sim s^{-1.8+p}$ so $\tau=1.8-p$. In all of the figures presented in this section the probability distribution data have been binned to produce clearer plots and aid in testing the prediction of a straight line.

Our results for $\alpha=0.22$ are shown in Fig. 1, which reveals that even for the largest system sizes the data do not fall on a single straight line. For small D_{av} ($0.2\leqslant D_{av}\leqslant 0.8$) the curves converge to a straight line with nonzero slope (consistent with $\tau\approx 1.70$). In contrast, for large D_{av} ($1.0\leqslant D_{av}\leqslant 1.8$) the curves reveal a much stronger L dependence with systems of size $L<256$, clearly much too small to identify

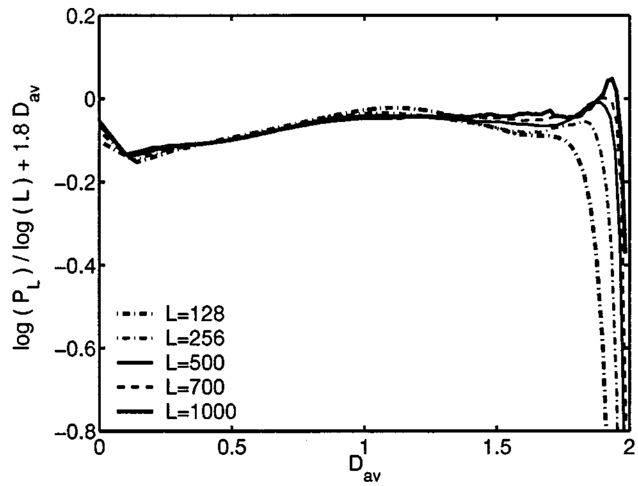

FIG. 1. Plot of $\log(P_L)/\log(L)+1.8D_{av}$ against D_{av} for $\alpha=0.22$ and for various system sizes $128\leqslant L\leqslant 1000$.

the asymptotic behavior. For larger L the curves do appear to be slowly converging to an approximately straight line. To unambiguously determine the slope of this line, systems of size larger than $L=1000$ would need to be studied, but on the basis of the available simulation results we can predict $\tau<1.8$ in this region. The failure to converge to a single straight line for the full range of D_{av} indicates that $P_L(s)$ is not dominated by a single power law, hence the system does not genuinely display the scale invariance associated with criticality.

Repeating our analysis for other dissipation levels reveals similar results. For example, in Fig. 2, we plot $\log(P_L)/\log(L)+1.8D_{av}$ versus D_{av} for $\alpha=0.16$ and a range of system sizes. Note that the curves cannot be fit by a single straight line, again casting doubt on the criticality of the system. There is evidence that the curves converge to an approximately straight line for small D_{av} (corresponding to a local slope exponent $\tau\approx 1.85$), while for larger D_{av} we again find stronger L dependence and for this α less indication of a straight line fit. We certainly do not believe that there is

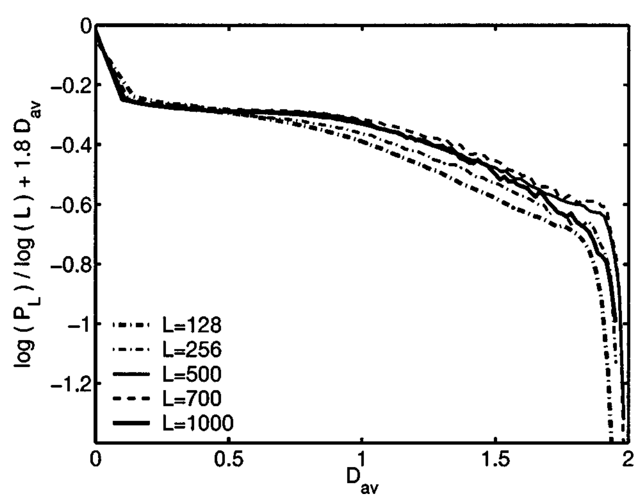

FIG. 2. Plot of $\log(P_L)/\log(L)+1.8D_{av}$ against D_{av} for $\alpha=0.16$ and the range of system sizes indicated.

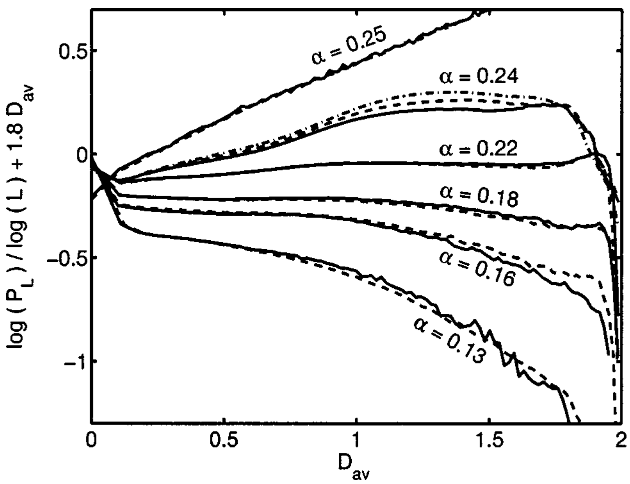

FIG. 3. Plot of $\log(P_L)/\log(L)+1.8D_{av}$ against D_{av} for various dissipation levels and a range of system sizes. For each choice of α the largest system size shown is indicated by a solid line. Specifically, the system sizes shown are $L=500$ and $L=700$ for $\alpha=0.25$, 0.22, and 0.18; $L=700$ (dot-dashed line), $L=1000$, and $L=1400$ for $\alpha=0.24$; $L=700$ and $L=1000$ for $\alpha=0.16$; and $L=256$ and $L=500$ for $\alpha=0.13$.

evidence of the curves converging to a single horizontal line for the full range of D_{av} on the basis of simulations up to size $L=1000$ [14], contrary to the claims of LP.

In Fig. 3 we plot $\log(P_L)/\log(L)+1.8D_{av}$ against D_{av} for various α values (several system sizes in each case) to highlight the α dependence of the probability distributions. In the conservative case, which is accepted to be critical, we find a single straight line fit for all $D_{av}>0$ with $\tau\approx1.25$. In all the dissipative cases there is clear nonlinear behavior for the system sizes studied, if the prediction of a universal slope exponent is correct all these curves must become straight and parallel as L is further increased. LP proposed for small $\alpha(\leqslant0.20)$ that although the distributions decrease from left to right for small system sizes, they will converge to a horizontal line as the system size is increased. This assertion holds to some extent for the smaller system sizes considered in Ref. [7], so as L is increased from 32 to 256 the curves could conceivably be converging towards a horizontal line. However, this pattern does not continue for the larger system sizes we have considered where the curves do not appear to converge towards a horizontal line, as clearly seen for the lower two systems shown in Fig. 3. For α values in the range $0.18\leqslant\alpha\leqslant0.22$ the deviation from linearity is the smallest, and the curves are nearest to being horizontal. LP concentrated primarily on α values in this range, which may explain why they concluded that τ was universal with $\tau\approx1.8$. However, the lines are not straight as seen clearly in Fig. 1, and there is a systematic change with increasing α, indicating an α dependence in the results, contrary to the proposal of universality. For larger α, close to the conservative limit, there is a strong L dependence in the results, implying that large systems need to be simulated in order to accurately identify asymptotic behavior. Our results for $\alpha=0.24$ shown in Fig. 3 suggest that the multiscaled probability distribution function approaches a predominantly increasing function from left to right so that the "average slope exponent" is less than than 1.8 (i.e., if we try to approximate the curves by a single straight line, the slope exponent for that line is less than 1.8).

Thus in conclusion we believe that there is no evidence for universality of the event size distribution functions. In the nonconservative regime the data cannot be fit by a single straight line so that the systems do not display scale invariance. Hence one cannot determine a single-slope exponent τ. However, if one looks at the overall shape of the curves to define an average slope exponent, $\overline{\tau}$, say, then one finds $\overline{\tau}>1.8$ for small α (corresponding to curves decreasing from left to right in Fig. 3), $\overline{\tau}\approx1.8$ for $\alpha\approx0.20$, and $\overline{\tau}<1.8$ for α close to the conservative limit (corresponding to curves increasing from left to right in Fig. 3). The systematic change in $\overline{\tau}$ with α, or if one prefers the α dependence of the slopes of $\log(P_L)/\log(L)$, is fully consistent with nonuniversality. We believe that the predictions of universality by LP are due in part to not considering large enough system sizes, and in part to the large scales used in their figures which make observing small deviations in slope particularly difficult.

III. THE BEHAVIOR OF $D_{\max}$

In this section we address how the dimension of the largest avalanche, $D_{\max}$, varies with system size for a given choice of dissipation level. Recall, LP proposed that $D_{\max}\to2$ as $L\to\infty$ for the range of α values they considered. In order to aid our analysis we concentrate on the cumulative distribution function

$$F_L(s)=\sum_{s'\geqslant s}P_L(s') \tag{3.1}$$

in this study. Note that if $P_L(s)\sim s^{-\tau}$, then $F_L(s)\sim s^{(1-\tau)}$ so that the slope exponents of the two distributions differ by one, however the largest measured event is the same in both distributions.

The dimension of an earthquake of size s in a system of size L is $D_{av}=\log(s)/\log(L)$. The dimension of the largest avalanche, $D_{\max}$, is formally defined as

$$D_{\max}=\max(D_{av}), \tag{3.2}$$

where the maximum is taken over a sufficiently large number of avalanches. In practice, one considers a multiscaling plot of $\log(F_L)/\log(L)$ versus D_{av} and identifies an effective $D_{\max}$ as the value of D_{av} at which $\log(F_L)/\log(L)$ takes some predetermined value. This allows a fairer comparison of $D_{\max}$ for different system sizes when using finite duration simulation studies. An example of such a plot is shown in Fig. 4 for the case $\alpha=0.13$. We see that $D_{\max}$ increases when L is increased from $L=128$ to 256, but upon further increasing system size $D_{\max}$ decreases. This indicates that $D_{\max}$ does not simply increase towards 2 as L increases, rather there is a crossover system size $L_{\max}^X(\alpha)$ above which $D_{\max}$ decreases with increasing L. For $\alpha=0.13$ it is clear from Fig. 4 that $128<L_{\max}^X(0.13)<384$. It is known that when one starts to organize the OFC model from random data there is a significant transient phase before the organized stationary state is

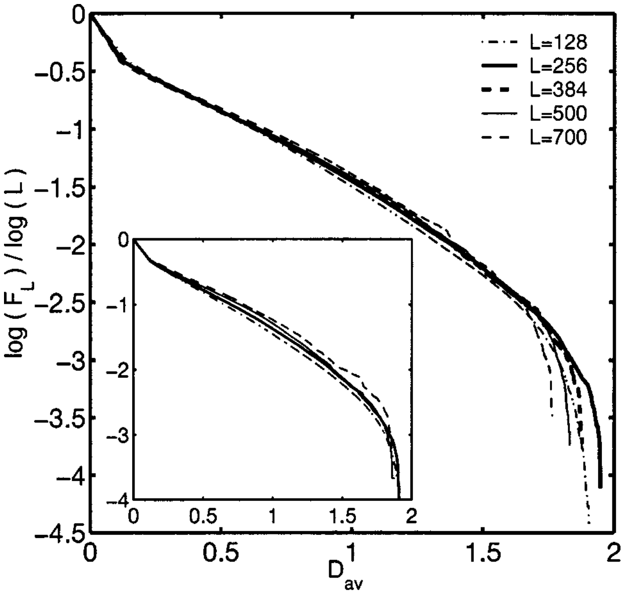

FIG. 4. Multiscaling plot of $\log(F_L)/\log(L)$ against D_{av} for $\alpha=0.13$ and various system sizes are indicated. The main plot shows the results of quadruple precision simulations, while the inset shows the corresponding results using double precision (for clarity the $L=384$ system is not shown in the inset).

reached [8]. During this transient phase the largest event size may continue to grow until the stationary state is reached, thus one must be careful to ensure that the decrease of $D_{\max}$ shown in Fig. 4 is not due to transient effects. In all cases we have run a minimum of 2×10^9 avalanches prior to collecting data, and checked that subsequent distributions overlay one another. For example, in Fig. 5 we show three consecutive runs of 6×10^9 avalanches for $L=256$ and $L=500$ which clearly demonstrates that the decrease of $D_{\max}$ is not a transient effect.

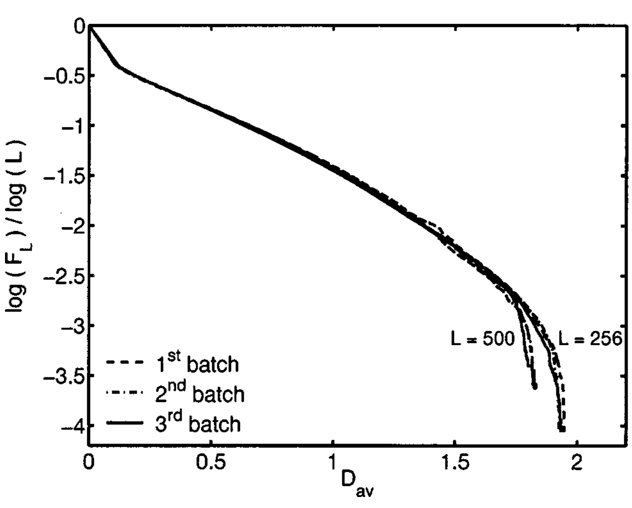

FIG. 5. Multiscaling plot for $\alpha=0.13$ and system sizes $L=256$ and $L=500$. In both cases three consecutive batches of 6×10^9 avalanches collected after the organizing phase are shown. The agreement between batches demonstrates that transient effects are not responsible for the observed decrease in $D_{\max}$.

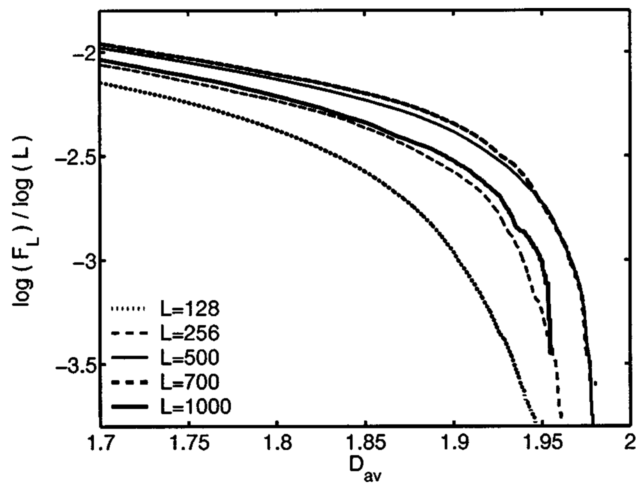

FIG. 6. Multiscaling plot of $\log(F_L)/\log(L)$ against D_{av} for $\alpha=0.16$ and for various system sizes $128\leqslant L\leqslant1000$.

In light of recent work some consideration should be given to the effect of the limited floating point precision used in the simulations. In particular, it has been shown that when using finite precision a series of small avalanches may be mistaken for a single larger avalanche, this effect can lead to nonrepresentative behavior for small α and large enough L [15,16]. Throughout our study we have used quadruple precision arithmetic which we believe is sufficiently accurate for the system sizes and dissipation levels considered [17]. The case $\alpha=0.13$, $L=700$ is borderline so that we cannot be sure that the result of Fig. 4 would be reproduced for all choices of initial data, however, this does not affect our conclusion above. The effect of limited precision is shown in the inset of Fig. 4, in which the study has been repeated using double precision arithmetic. Note that for $L\geqslant500$ one finds an excess of large avalanches and a different prediction for the slope exponent, thus the results of our study are sensitive to the level of precision used in the simulations, implying that one should be wary of predictions made on the basis of low precision simulations.

Returning to our analysis of $D_{\max}$, we find that the qualitative behavior for $\alpha=0.13$ is repeated for other choices of dissipation levels. In Fig. 6 we show the results for $\alpha=0.16$, here for clarity only the tails of the distributions are plotted. Once again $D_{\max}$ initially increases with system size [until $L=L^X_{\max}(0.16)$] and then decreases, with $500<L^X_{\max}(0.16)<1000$. In Fig. 7 we show distributions for $L=500$ and $L=1000$ at two consecutive periods after the initial organization phase, showing that these results are not due to transient effects.

In general, we find that as α increases $L^X_{\max}(\alpha)$ also increases so that large systems need to be simulated in order to observe this behavior. In their study LP considered $\alpha\geqslant0.16$ and $L\leqslant512$, which explains why they did not observe the existence of $L^X_{\max}(\alpha)$. From our investigation (considering $L\leqslant1000$) we have explicitly observed this behavior for $\alpha<0.18$ and believe that similar results would be found for larger choices of $\alpha<0.25$ if sufficiently large system sizes could be simulated. However, due to the long transient periods required to reach the organized state in large systems [8],

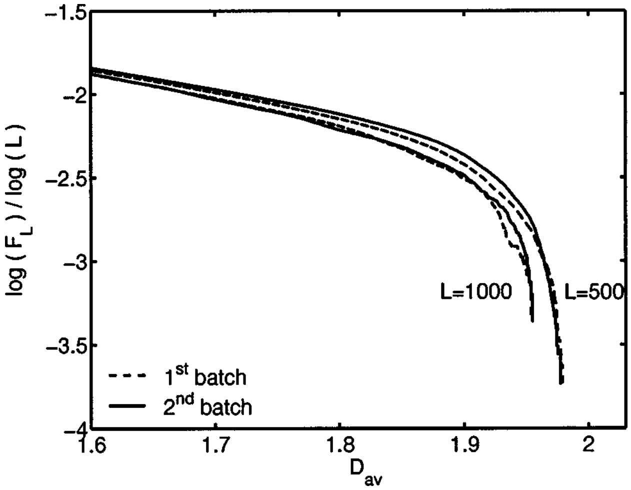

FIG. 7. Multiscaling plot for $\alpha=0.16$ and system sizes $L=500$ and $L=1000$, confirming that the simulations have been run past the transient phase. In both cases two consecutive batches of 12×10^9 avalanches collected after the organizing phase are shown.

coupled with increasing average avalanche sizes when α is increased, this is computationally prohibitive at present.

IV. DISCUSSION AND CONCLUSIONS

In this paper we have used a multiscaling approach to numerically analyze several features of the OFC model. Our two main findings are as follows: first, we find that multiscaled plots of the event size distribution cannot be fit by a single straight line, except in the conservative case $\alpha=0.25$. This suggests that the scale invariance associated with criticality is only present in the conservative limit. Furthermore, there is no evidence that one can fit a universal slope exponent for these distributions, as is clearly seen in Fig. 3. It is important to recognize that the concept of universality is a very strong one, and if found in the OFC model would be a significant result. Disproving possible universality in infinite-sized systems based on finite-size simulation results is difficult (since one can never be fully sure what would happen if systems of a magnitude or two larger could be simulated). However, we provide clear evidence that there is no justification for the claims of universality made in Ref. [7], with the available data strongly suggesting nonuniversality. We wish to stress that, provided the dissipation is not too large (restricting $\alpha\gtrsim0.18$ say), the deviation from linearity in the probability distributions is relatively small, so that the OFC model may still be valid for explaining the approximately power-law distributions observed over many decades in physical systems.

Second, for a fixed α we do not find that the dimension of the largest avalanche (D_{max}) increases towards 2 as the system size is increased, contrary to earlier indications [7]. Instead, we believe that there exists a crossover system size $L^X_{\mathrm{max}}(\alpha)$ such that once $L>L^X_{\mathrm{max}}(\alpha)$ the maximum avalanche dimension decreases as L increases. As explained below, this behavior, in which D_{max} initially increases and then decreases as L is increased, is fully consistent with the notion that the OFC model is almost critical [18] rather than truly critical. Earlier studies have indicated that for fixed $\alpha<0.25$, as L is increased the overall system branching rate σ initially increases rapidly before leveling off at a limiting value $\sigma=\sigma_{\mathrm{lim}}(\alpha)<1$ [10,11]. The initial increase of D_{max} as L is increased is associated with the rapidly increasing branching rate allowing the possibility of proportionally larger dissipating events. When L is further increased D_{max} decreases, indicating that the relative size of the largest dissipating events with respect to the maximum total force allowed in the system decreases. This is precisely what one would expect if the system is not critical but is nearly so [with $L^X_{\mathrm{max}}(\alpha)$ increasing as $\sigma_{\mathrm{lim}}(\alpha)$ approaches 1]. It is appropriate to note that one does not require $\lim_{L\to\infty}D_{\mathrm{max}}=2$ for criticality. Provided D_{max} approaches some nonzero limit, one could still have criticality with a somewhat reduced dimension for the largest dissipating event. However, the nonlinearity in the probability distributions combined with the observed decrease of D_{max} are all consistent with the hypothesis that the nonconservative OFC model is not critical.

ACKNOWLEDGMENT

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